Indian Statistical Institute, Bangalore
B. Math II, First Semester, 2024-25
Final Examination, Introduction to Statistical Inference

11.11.24 **Maximum Score 100** **Duration: 3 Hours**

1. (10) Let $X_1, \cdots, X_n$ be iid Bernoulli(p) random variables. State with reason, which of the following is/are sufficient statistics for the parameter p.
   (a) $\sum_{i=1}^n X_i$
   (b) $n - \sum_{i=1}^n X_i$
   (c) $\sum_{i=1}^n X_i - np$
   (d) X_1
   (e) $(X_1, \sum_{i=2}^n X_i)$

2. (10) Determine which of the following are full exponential families. If they are not, give reason. If they are, then specify the canonical parameter, natural parameter space and natural sufficient statistic when a sample $X_1, \cdots, X_n$ is taken from the family.
   (a) $f(x,\theta) = \begin{cases} \frac{1}{\lambda} e^{-\frac{x-\mu}{\lambda}}, & x > -\mu \\ 0 & \text{otherwise} \end{cases}$
   (b) $\mathcal{N}(\mu, \mu^2)$
   (c) $\mathcal{N}(\mu, \sigma^2)$

3. (12) Prove the following result. Let U be the class of all unbiased estimators T of $\theta \in \Theta$ with $E_\theta(T^2) < \infty \, \forall \theta$, and suppose that U is non–empty. Let U_0 be the set of all unbiased estimates of zero, that is, $U_0 = \{\nu : E_\theta(\nu) = 0, E_\theta(\nu^2) < \infty \, \forall \theta \in \Theta\}$. Then $T_0 \in U$ is UMVUE iff $E_\theta(\nu T_0) = 0 \, \forall \theta \in \Theta \, \forall \nu \in U_0$.

4. (3+10) Consider the random vector $Z = \begin{pmatrix} X \\ Y \end{pmatrix}$ where the dimension of X is 2×1 and that of Y is 1×1.
$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 & 4 & 1 \\ 4 & 12 & 2 \\ 1 & 2 & 2 \end{pmatrix} \right)$$
   Find the marginal distribution of Y and the conditional distribution of Y given $X = x$.

5. (2+5+5+3+5+5)Suppose $X_1, \cdots, X_n$ are iid Poisson(λ)
   (a) Find an unbiased estmator for $\theta = e^{-\lambda}$ based on X_1.
   (b) Use Rao Blackwell theorem to find an unbiased estimator $\hat{\theta}_U$ of θ based on $\bar{X}$, that has lower variance then the estimator in part (a).
   (c) Find the variance of $\hat{\theta}_U$.
   (d) Find the MLE $\hat{\theta}_{MLE}$ of θ.
   (e) Find the bias and variance of $\hat{\theta}_{MLE}$.
   (f) Show that the limit, as $n \to \infty$, of Var($\sqrt{n}\hat{\theta}_{MLE}$) and Var($\sqrt{n}\hat{\theta}_U$) are the same.

   Note: For independent Poisson random variables with parameters a and b, the distribution of the sum is Poisson $(a + b)$.

6. (10) Suppose $X_1, \cdots, X_n$ are iid $\mathcal{N}(0, \sigma^2)$ random variables. Construct two $(1 - \alpha)$ confidence interval for σ based on the following pivots. Which of these is better and why?
$$\frac{\bar{X}}{\sigma} \quad \text{and} \quad \frac{\sum_{i=1}^n X_i^2}{\sigma^2}$$

7. (14) Let $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_m$ be two independent samples from normal distributions with equal variances and means μ_1 and μ_2 respectively. Find the generalized likelihood ratio test for testing $\mu_1 = \mu_2$ at level α. Find the test statistics and its distribution under the null hypothesis.

8. (6) Suppose we have obtained an unbiased test of size 4%. Answer the following with reasons.

   (a) Is this a level 5% test?

   (b) Can the power of this test be 3% under the alternative?

   (c) If, with a particular dataset, the p-value of this test is 3%, shall we reject the null hypothesis?